

A fixed point theorem for Kannan-type maps in metric spaces

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Abstract. We prove a generalization of Kannan's fixed point theorem, based on a recent result of Vittorino Pata.

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1. Introduction

Our starting point is Kannan's result in metric fixed point theory involving contractive type mappings which are not necessarily continuous [2]. It has been shown in [5] that Kannan's theorem is independent of the famous Banach contraction principle (see, e.g. [1]), and that it also characterizes the metric completeness concept [6].

Definition 1.1. Let (X, d) be a metric space. Let us call $T : X \rightarrow X$ a *Kannan map* if there exists some $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \leq \frac{\lambda}{2} \{d(x, Tx) + d(y, Ty)\} \quad (1.1)$$

for all $x, y \in X$.

For complete metric spaces, Kannan proved the following:

Theorem 1.2. ([2]) *If (X, d) is a complete metric space, and if T is a Kannan map on X , then there exists a unique $x \in X$ such that $Tx = x$.*

And Subrahmanyam (in [6]) has proved the counterpart by showing that if all the Kannan maps on a metric space have fixed points then that space must necessarily be complete.

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2. Generalization of Kannan's fixed point theorem

As in [4], from this point onwards let (X, d) stand for a complete metric space. Let us select arbitrarily a point $x_0 \in X$, and call it the "zero" of X . We denote

$$\|x\| := d(x, x_0), \forall x \in X.$$

Let $\Lambda \geq 0$, $\alpha \geq 1$, and $\beta \in [0, \alpha]$ be fixed constants, and let $\psi : [0, 1] \rightarrow [0, \infty)$ denote a preassigned increasing function that vanishes (with continuity) at zero. Then, for a map $T : X \rightarrow X$, Pata shows that the following theorem holds.

Theorem 2.1. ([4]) *If the inequality*

$$d(Tx, Ty) \leq (1 - \varepsilon)d(x, y) + \Lambda \varepsilon^\alpha \psi(\varepsilon)[1 + \|x\| + \|y\|]^\beta \quad (2.1)$$

is satisfied for every $\varepsilon \in [0, 1]$ and every $x, y \in X$, then T possesses a unique fixed point $x_ = Tx_*$ ($x_* \in X$).*

Motivated by this generalization of the Banach fixed point theorem, we can come up with an analogous generalized form of Theorem 1.2.

2.1. The main theorem

With all the other conditions and notations remaining the same except for a more general $\beta \geq 0$, our goal is to prove the following:

Theorem 2.2. *If the inequality*

$$\begin{aligned} & d(Tx, Ty) \\ & \leq \frac{1 - \varepsilon}{2} \{d(x, Tx) + d(y, Ty)\} + \Lambda \varepsilon^\alpha \psi(\varepsilon) [1 + \|x\| + \|Tx\| + \|y\| + \|Ty\|]^\beta \end{aligned} \quad (2.2)$$

is satisfied $\forall \varepsilon \in [0, 1]$ and $\forall x, y \in X$, then T possesses a unique fixed point $x^ = Tx^*$ ($x^* \in X$).*

Remark 2.3. Since we can always redefine Λ to keep (2.2) valid no matter what initial $x_0 \in X$ we choose, we are in no way restricting ourselves by taking that zero instead of a generic $x \in X$ [4].

2.2. Proofs

2.2.1. Uniqueness of x^* . We claim first that such an x^* , if it exists, is unique. To see that this is the case, let, if possible, $\exists x^*, y^* \in X$ such that

$$\begin{aligned} x^* &= Tx^*, \\ y^* &= Ty^*, \\ \text{and } x^* &\neq y^*. \end{aligned}$$

Then (2.2) implies, $\forall \varepsilon \in [0, 1]$,

$$d(x^*, y^*) \leq \Lambda \varepsilon^\alpha \psi(\varepsilon) [1 + \|x^*\| + \|Tx^*\| + \|y^*\| + \|Ty^*\|]^\beta.$$

In particular, $\varepsilon = 0$ gives us

$$\begin{aligned} d(x^*, y^*) &\leq 0 \\ \implies x^* &= y^*, \end{aligned}$$

which is a contradiction.

2.2.2. Existence of x^* . We now bring into play the two sequences

$$\begin{aligned} x_n &= Tx_{n-1} = T^n x_0 \\ \text{and } c_n &= \|x_n\|, n = 1, 2, 3, \dots \end{aligned}$$

But before we proceed any further, we will need the following:

Lemma 2.4. $\{c_n\}$ is bounded.

Proof. From (2.2), considering again the case of $\varepsilon = 0$, we see that for $n = 1, 2, 3, \dots$,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \frac{1}{2} \{d(x_{n+1}, x_n) + d(x_n, x_{n-1})\} \\ \implies d(x_{n+1}, x_n) &\leq d(x_n, x_{n-1}) \\ &\vdots \\ &\leq d(x_1, x_0) = c_1. \end{aligned} \tag{2.3}$$

Now, $\forall n \in \mathbb{N}$,

$$\begin{aligned} c_n &= d(x_n, x_0) \\ &\leq d(x_n, x_1) + d(x_1, x_0) \\ &= d(x_n, x_1) + c_1 \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_1) + c_1 \\ &\leq c_1 + d(x_{n+1}, x_1) + c_1 && [\text{using (2.3)}] \\ &\leq d(Tx_n, Tx_0) + 2c_1 \\ &\leq \frac{1}{2} \{d(x_{n+1}, x_n) + d(x_1, x_0)\} + 2c_1 && [\text{using (2.2) with } \varepsilon = 0] \\ &\leq \frac{1}{2} (c_1 + c_1) + 2c_1 && [(2.3)] \\ &= 3c_1. \end{aligned}$$

And hence the lemma is proved. □

Next we strive to show that:

Lemma 2.5. $\{x_n\}$ is Cauchy.

Proof. In light of (2.2), for $n = 1, 2, 3, \dots$,

$$\begin{aligned}
 & d(x_{n+1}, x_n) \\
 &= d(Tx_n, Tx_{n-1}) \\
 &\leq \frac{1-\varepsilon}{2} \{d(x_{n+1}, x_n) + d(x_n, x_{n-1})\} \\
 &\quad + \Lambda \varepsilon^\alpha \psi(\varepsilon) [1 + \|x_{n+1}\| + \|x_n\| + \|x_{n-1}\|]^\beta \\
 &\leq \frac{1-\varepsilon}{2} \{d(x_{n+1}, x_n) + d(x_n, x_{n-1})\} + C \varepsilon^\alpha \psi(\varepsilon)
 \end{aligned}$$

for $C = \sup_{j \in \mathbb{N}} \Lambda(1 + 4c_j)^\beta < \infty$ (on account of Lemma 2.4). But then, $\forall \varepsilon \in (0, 1]$,

$$\begin{aligned}
 & d(x_{n+1}, x_n) \\
 &\leq \frac{1-\varepsilon}{1+\varepsilon} d(x_n, x_{n-1}) + \frac{2C\varepsilon^\alpha}{1+\varepsilon} \psi(\varepsilon) \\
 &\vdots \\
 &\leq \dots \\
 &\vdots \\
 &\leq \left(\frac{1-\varepsilon}{1+\varepsilon} \right)^n d(x_1, x_0) \\
 &\quad + \frac{2C\varepsilon^\alpha}{1+\varepsilon} \psi(\varepsilon) \left[1 + \frac{1-\varepsilon}{1+\varepsilon} + \dots + \left(\frac{1-\varepsilon}{1+\varepsilon} \right)^{n-1} \right] \\
 &\leq k^n d(x_1, x_0) \\
 &\quad + \frac{2C\varepsilon^\alpha}{1+\varepsilon} \psi(\varepsilon) (1 + k + \dots + k^{n-1}) \quad [\text{letting } k = \frac{1-\varepsilon}{1+\varepsilon} \geq 0] \\
 &\leq k^n d(x_1, x_0) \\
 &\quad + \frac{2C\varepsilon^\alpha}{1+\varepsilon} \psi(\varepsilon) (1 + k + \dots + k^{n-1} + \dots) \quad [\text{because } k \geq 0] \\
 &\leq k^n d(x_1, x_0) \\
 &\quad + \frac{2C\varepsilon^\alpha}{1+\varepsilon} \psi(\varepsilon) \frac{1}{1-k} \\
 &= k^n d(x_1, x_0) + C \varepsilon^{\alpha-1} \psi(\varepsilon) \quad [\text{putting } k = \frac{1-\varepsilon}{1+\varepsilon}]
 \end{aligned} \tag{2.4}$$

for all $n \in \mathbb{N}$.

At this point we note that if $\varepsilon \in (0, 1]$, then $k < 1$. Therefore, taking progressively lower values of ε that approach zero but never quite reach it, the R.H.S. of (2.4) can be made as small as one wishes it to be as $n \rightarrow \infty$. Indeed, since $C \varepsilon^{\alpha-1} \psi(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0+$, for an arbitrary $\eta > 0$, $\exists \varepsilon = \varepsilon(\eta) > 0$ such that $C \varepsilon^{\alpha-1} \psi(\varepsilon) < \frac{\eta}{2}$. Again, for this $\varepsilon (= \varepsilon(\eta))$, $\exists N \in \mathbb{N}$ such that

$k^n d(x_1, x_0) < \frac{\eta}{2} \forall n \geq N$ because $k^n d(x_1, x_0) \rightarrow 0$ as $n \rightarrow \infty$. Together that gives us

$$k^n d(x_1, x_0) + C\varepsilon^{\alpha-1}\psi(\varepsilon) < \frac{\eta}{2} + \frac{\eta}{2} = \eta, \forall n \geq N.$$

In other words,

$$d(x_n, x_{n+1}) \rightarrow 0 \text{ as } n \rightarrow \infty, \varepsilon \rightarrow 0+. \quad (2.5)$$

Hence, from (2.2), using the same $C = \sup_{j \in \mathbb{N}} \Lambda(1 + 4c_j)^\beta$, and letting $n \rightarrow \infty, \varepsilon \rightarrow 0+$,

$$\begin{aligned} & d(x_n, x_{n+p}) \\ &= d(Tx_{n-1}, Tx_{n+p-1}) \\ &\leq \frac{1-\varepsilon}{2} \{d(x_{n-1}, x_n) + d(x_{n+p-1}, x_{n+p})\} + C\varepsilon^\alpha\psi(\varepsilon) \\ &\rightarrow 0 \end{aligned} \quad [\text{using (2.5)}]$$

uniformly over $p = 1, 2, \dots$, which basically assures us that $\{x_n\}$ is Cauchy. \square

Equipped with (2.5) and taking into note the completeness of X , we can now safely guarantee the existence of some $x^* \in X$ to which $\{x_n\}$ converges.

Finally, all that remains to show is that:

2.2.3. x^* is a fixed point for T . For this we observe that, $\forall n \in \mathbb{N}$,

$$\begin{aligned} & d(Tx^*, x^*) \\ &\leq d(Tx^*, x_{n+1}) + d(x_{n+1}, x^*) \\ &= d(Tx^*, Tx_n) + d(x_{n+1}, x^*) \\ &\leq \frac{1}{2} \{d(Tx^*, x^*) + d(Tx_n, x_n)\} + d(x_{n+1}, x^*) \\ &[\text{using (2.2) with } \varepsilon = 0 \text{ again}] \\ &\implies \frac{1}{2}d(Tx^*, x^*) \leq \frac{1}{2}d(x_n, x_{n+1}) + d(x_{n+1}, x^*) \end{aligned} \quad (2.6)$$

As $n \rightarrow \infty$ (and $\varepsilon \rightarrow 0+$), we know that:

$$\begin{aligned} d(x_n, x_{n+1}) &\rightarrow 0 && [\text{from (2.5)}]; \\ d(x_{n+1}, x^*) &\rightarrow 0 && [\text{since } x_n \rightarrow x^*]. \end{aligned}$$

So (2.6) actually gives us that

$$\begin{aligned} d(x^*, Tx^*) &\leq 0 \\ \implies Tx^* &= x^*, \end{aligned}$$

which is the required result.

3. Comparison with Kannan's Original Result

The requirements of Theorem 2.2 are indeed weaker than those of Kannan's theorem. To see that, let us start from (1.1) with $\lambda \in (0, 1)$ (barring the trivial case where $\lambda = 0$).

We have, $\forall \varepsilon \in [0, 1]$,

$$\begin{aligned}
 & d(Tx, Ty) \\
 & \leq \frac{\lambda}{2} \{d(x, Tx) + d(y, Ty)\} \\
 & \leq \frac{1-\varepsilon}{2} \{d(x, Tx) + d(y, Ty)\} + \frac{\lambda + \varepsilon - 1}{2} \{d(x, Tx) + d(y, Ty)\} \\
 & = \frac{1-\varepsilon}{2} \{d(x, Tx) + d(y, Ty)\} \\
 & \quad + \frac{\lambda}{2} \left(1 + \frac{\varepsilon - 1}{\lambda}\right) \{d(x, Tx) + d(y, Ty)\} \\
 & \leq \frac{1-\varepsilon}{2} \{d(x, Tx) + d(y, Ty)\} + \frac{\lambda}{2} (1 + \overline{\varepsilon - 1})^{\frac{1}{\lambda}} \{d(x, Tx) + d(y, Ty)\} \\
 & \quad [\text{using Bernoulli's Inequality, since } \varepsilon - 1 \geq -1 \text{ \& } \frac{1}{\lambda} > 1] \\
 & \leq \frac{1-\varepsilon}{2} \{d(x, Tx) + d(y, Ty)\} \\
 & \quad + \frac{\lambda}{2} \varepsilon^{\frac{1}{\lambda}} [1 + \{d(x, x_0) + d(x_0, Tx)\} + \{d(y, x_0) + d(x_0, Ty)\}] \\
 & = \frac{1-\varepsilon}{2} \{d(x, Tx) + d(y, Ty)\} + \frac{\lambda}{2} \varepsilon^{1+\gamma} [1 + \|x\| + \|Tx\| + \|y\| + \|Ty\|] \\
 & \quad [\text{taking } 1 < \frac{1}{\lambda} = 1 + \gamma \text{ for some } \gamma > 0] \\
 & \leq \frac{1-\varepsilon}{2} \{d(x, Tx) + d(y, Ty)\} + \frac{\lambda}{2} \varepsilon \varepsilon^\gamma [1 + \|x\| + \|Tx\| + \|y\| + \|Ty\|].
 \end{aligned} \tag{3.1}$$

Then a quick comparison between (2.2) and (3.1) with $\psi(\varepsilon) = \varepsilon^\gamma$ ($\gamma > 0$) provides us with what we need.

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